

CIRCLE CORRESPONDENCE C^* -ALGEBRAS

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ABSTRACT. We investigate Cuntz-Pimsner C^* -algebras associated with certain correspondences of the unit circle \mathbb{T} . We analyze these C^* -algebras by analogy with irrational rotation algebras A_θ and Cuntz algebras \mathcal{O}_n . We construct a Rieffel type projection, study the fixed point algebras of certain actions of finite groups, and calculate the entropy of a certain endomorphism. We also study the induced map of the dual action of the gauge action on K -groups.

1. INTRODUCTION

In [KW], Kajiwara and Watatani introduced a C^* -algebra \mathcal{O}_R associated with a rational function R as a Cuntz-Pimsner algebra of a Hilbert bimodule over $A = C(J_R)$, which is the algebra of continuous functions on Julia set J_R of R . They proved that if the degree of R is at least two, then a C^* -algebra \mathcal{O}_R is a *Kirchberg algebra* (purely infinite, simple, nuclear, separable C^* -algebra) satisfying UCT. In this framework, one of the most important issues is to observe relations between C^* -algebras \mathcal{O}_R and complex dynamical systems. But it is also important to examine the properties for elementary rational functions.

The Cuntz algebras and the irrational rotation algebras have been examined by many authors. These algebras are simple and universal C^* -algebras with certain commutation relations. They have their own properties, which are not easily observed in more general C^* -algebras.

In this paper, we deal with a C^* -algebra $\mathcal{O}_{(m,n)}(\mathbb{T})$ which is generated by two elements z and S_1 with commutation relations $z^n S_1 = S_1 z^m$, $\sum_{i=0}^{n-1} z^i S_1 S_1^* z^{-i} = 1$. This C^* -algebra include the above C^* -algebra \mathcal{O}_R associated with the elementary rational function $R(z) = z^n$ on $J_R = \mathbb{T}$. Our algebras have appeared in several papers (for example, a groupoid C^* -algebra [Dea] and a topological graph C^* -algebra [Kat]). The main purpose of the present paper is to examine specific properties of our algebras like Cuntz algebras \mathcal{O}_n and irrational rotation algebras A_θ . In fact, our algebras contain Cuntz algebras as subalgebras and are generated by two elements like the C^* -algebra A_θ .

This paper is organized as follows. In section 2, we give some preliminaries. In section 3, we construct a projection on matrix algebra over $\mathcal{O}_{(m,n)}(\mathbb{T})$ for $(m,n) = (1,2)$. This construction is similar to that of the C^* -algebra A_θ presented by Rieffel [Rie]. In section 4, we discuss the C^* -subalgebras of $\mathcal{O}_{(m,n)}(\mathbb{T})$. In the case of the C^* -algebra A_θ , for any positive integer k , $A_{k\theta}$ is naturally a C^* -subalgebra of A_θ . We treat an analog of this problem for $\mathcal{O}_{(m,n)}(\mathbb{T})$. We apply this result to obtain the fixed point algebra of a cyclic group action on $\mathcal{O}_{(m,n)}(\mathbb{T})$. Moreover we determine the fixed point of a symmetry action. In section 5, we calculate the entropy of a certain endomorphism that seems like Cuntz's

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canonical endomorphism. In section 6, we consider the gauge action and its dual action of $\mathcal{O}_{(m,n)}(\mathbb{T})$. In particular, we investigate the induced map of the dual action of the gauge action on K -groups. As a corollary, we obtain another computation of K -groups.

2. PRELIMINARIES

In this section, we recall the Cuntz-Pimsner algebras and introduce the C^* -algebras $\mathcal{O}_{(m,n)}(\mathbb{T})$. A (*right*) *Hilbert A -module* X is a Banach space (whose norm is $\|\cdot\|_2$) with a right action of a C^* -algebra A and an A -valued inner product $\langle \cdot, \cdot \rangle$ satisfying:

- (1) $\langle f, ga \rangle = \langle f, g \rangle a$
- (2) $\langle f, g \rangle^* = \langle g, f \rangle$
- (3) $\langle f, f \rangle \geq 0$ and $\|f\|_2 = \|\langle f, f \rangle\|^{1/2}$

for $f, g \in X$ and $a \in A$. The Hilbert A -module is *finitely generated* if there exists $\{u_i\}_{i=1}^n \subset X$ such that $f = \sum_{i=1}^n u_i \langle u_i, f \rangle$ for any $f \in X$. A Hilbert A -module X is *full* if $\overline{\text{span}}\{\langle f, g \rangle | f, g \in X\} = A$. We only use finitely generated full Hilbert bimodules. For a Hilbert A -module X , we denote by $L(X)$ the C^* -algebra of all adjointable operators on X . For $f, g \in X$, the rank-one operator $\theta_{f,g} \in L(X)$ is defined by $\theta_{f,g}(h) = f \langle g, h \rangle$ for $h \in X$. Set $K(X) = \overline{\text{span}}\{\theta_{f,g} | f, g \in X\}$. If X is finitely generated by $\{u_i\}_{i=1}^n$ and $\langle u_i, u_j \rangle = \delta_{i,j} 1$, then $L(X) = K(X)$ and $K(X)$ is isomorphic to $M_n(A)$ via $\{u_i\}_{i=1}^n$. We recall the Cuntz-Pimsner algebras. Let A be a C^* -algebra and X be a full Hilbert A -module that is finitely generated by $\{u_i\}_{i=1}^n$ and let $\phi : A \rightarrow L(X)$ be a faithful $*$ -homomorphism. We shall define a left action by $a \cdot f := \phi(a)f$ for $a \in A, f \in X$. Then the *Cuntz-Pimsner algebra* \mathcal{O}_X is the universal C^* -algebra generated by A and $\{S_f | f \in X\}$ with the following relation: for $a, b \in A, f, g \in X, \alpha, \beta \in \mathbb{C}$,

$$S_{\alpha f + \beta g} = \alpha S_f + \beta S_g, \quad a \cdot S_f \cdot b = S_{a \cdot f \cdot b}, \quad S_f^* S_g = \langle f, g \rangle, \quad \sum_{i=1}^n S_{u_i} S_{u_i}^* = 1.$$

Let us define the C^* -algebra $\mathcal{O}_{(m,n)}(\mathbb{T})$ for $m, n \in \mathbb{N}$ using the Cuntz-Pimsner construction. Set $\Omega_{(m,n)} = \{(t^m, t^n) \in \mathbb{T} \times \mathbb{T} | t \in \mathbb{T}\}$. Set $X_{(m,n)} = C(\Omega_{(m,n)})$. Then $X_{(m,n)}$ is a $C(\mathbb{T})$ - $C(\mathbb{T})$ bimodule by

$$(a \cdot f \cdot b)(x, y) = a(x) f(x, y) b(y)$$

for $a, b \in C(\mathbb{T}), f \in X_{(m,n)}, (x, y) \in \Omega_{(m,n)}$. We introduce a $C(\mathbb{T})$ -valued inner product $\langle \cdot, \cdot \rangle$ on $X_{(m,n)}$ by

$$\langle f, g \rangle(y) = \sum_{\{x \in \mathbb{T} | (x, y) \in \Omega_{(m,n)}\}} \overline{f(x, y)} g(x, y)$$

for $f, g \in X_{(m,n)}$ and $y \in \mathbb{T}$. Put $\|f\|_2 = \|\langle f, f \rangle\|_\infty^{1/2}$. Then $X_{(m,n)}$ is a full Hilbert bimodule over $C(\mathbb{T})$ without completion (see Corollary 2.3 of [KW]). Let us denote the greatest common divisor of $m, n \in \mathbb{N}$ by $\gcd(m, n)$. In this case, the Hilbert module $X_{(m,n)}$ is a finitely generated by $u_i(x, y) = \frac{1}{\sqrt{n_0}} x^{i-1}$ ($i = 1, \dots, n_0 := n/\gcd(m, n)$) and $\{u_i\}_{i=1}^{n_0}$ satisfies $\langle u_i, u_j \rangle = \delta_{i,j} 1$. Moreover $X_{(m,n)}$ is isomorphic to $X_{(m_0, n_0)}$ as a Hilbert module where $m_0 = m/\gcd(m, n)$. **Thus we always assume $\gcd(m, n) = 1$.**

We denote $S_i = S_{u_i}$ for simplicity. Let z be an element of $C(\mathbb{T})$ defined by $z(x) = x$ for any $x \in \mathbb{T}$. Then $\mathcal{O}_{(m,n)}(\mathbb{T})$ is the universal C^* -algebra generated by a (full-spectrum) unitary z and isometries S_1, \dots, S_n with the relation

$$zS_i = S_{i+1} \ (1 \leq i \leq n-1), \quad zS_n = S_1 z^m, \quad \sum_{i=1}^n S_i S_i^* = 1.$$

We will often use this relation. From this relation, we notice that $\mathcal{O}_{(m,n)}(\mathbb{T})$ is generated by the two elements z and S_1 .

For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the set $\mathcal{W}_n^{(k)}$ of k -triples by $\mathcal{W}_n^{(0)} = \{\emptyset\}$ and $\mathcal{W}_n^{(k)} = \{(i_1, i_2, \dots, i_k) | i_j \in \{1, \dots, n\}\}$. Set $\mathcal{W}_n = \bigcup_{k=0}^{\infty} \mathcal{W}_n^{(k)}$. For $\mu = (i_1, \dots, i_k) \in \mathcal{W}_n$, we denote its length k by $|\mu|$ and set $S_\mu = S_{i_1} S_{i_2} \dots S_{i_k}$. Note that $|\emptyset| = 0, S_\emptyset = 1$. For $\mu = (i_1, \dots, i_k), \nu = (j_1, \dots, j_l) \in \mathcal{W}_n$, we define their product $\mu\nu \in \mathcal{W}_n$ by $\mu\nu = (i_1, \dots, i_k, j_1, \dots, j_l)$. Using these notations, since $X_{(m,n)}$ is finitely generated, $\mathcal{O}_{(m,n)}(\mathbb{T})$ is presented by

$$\mathcal{O}_{(m,n)}(\mathbb{T}) = \overline{\text{span}} \left\{ S_\mu z^k S_\nu^* \mid k \in \mathbb{Z}, \mu, \nu \in \mathcal{W}_n \right\}.$$

There exists an action $\alpha : \mathbb{T} \ni t \mapsto \alpha_t \in \text{Aut}(\mathcal{O}_{(m,n)}(\mathbb{T}))$ with $\alpha_t(S_f) = tS_f$ that is called the *gauge action*. The fixed point algebra $\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha$ by the gauge action is the (m,n) -type Bunce-Deddens algebra $\mathcal{B}_{(m,n)} := \varinjlim_{k \geq 0} \{M_{n^k}(C(\mathbb{T})), \phi_k\}$ where $\phi_k : M_{n^k}(C(\mathbb{T})) \rightarrow M_{n^{k+1}}(C(\mathbb{T}))$ is defined by

$$\phi_k(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \times_n & \vdots \\ 0 & \cdots & a \end{pmatrix} \quad (a \in M_n(\mathbb{C})), \quad \phi_k \begin{pmatrix} 0 & z \\ 1_{n^{k-1}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & z^m \\ 1_{n^{k+1-1}} & 0 \end{pmatrix}.$$

The element $S_\mu z^k S_\nu^* \in \mathcal{O}_{(m,n)}(\mathbb{T})^\alpha$ corresponds to $z^k \otimes e_{\mu\nu} = z^k \otimes e_{\mu_1\nu_1} \otimes \dots \otimes e_{\mu_{|\mu|}\nu_{|\nu|}}$ where e_{ij} is the (i,j) -matrix unit.

We shall discuss about some properties on $\mathcal{O}_{(m,n)}(\mathbb{T})$. Recently, Katsura showed that some his algebras, so-called topological graph C^* -algebras, are Kirchberg algebras satisfying UCT. Our algebras $\mathcal{O}_{(m,n)}(\mathbb{T})$ are contained in his algebras. Katsura also computed these K -groups in Appendix A of [Kat3].

Theorem 2.1 (Katsura [Kat3]). *Suppose that $\gcd(m, n) = 1$.*

- (1) *For $m \geq 1, n \geq 2$, $\mathcal{O}_{(m,n)}(\mathbb{T})$ are Kirchberg algebras satisfying UCT.*
- (2) (a) *For $n \geq 2$, $K_0(\mathcal{O}_{(1,n)}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}$, $K_1(\mathcal{O}_{(1,n)}(\mathbb{T})) = \mathbb{Z}$.*
 (b) *For $m \geq 2$, $K_0(\mathcal{O}_{(m,1)}(\mathbb{T})) = \mathbb{Z}$, $K_1(\mathcal{O}_{(m,1)}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{m-1}$.*
 (c) *For $m, n \geq 2$, $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})) = \mathbb{Z}_{n-1}$, $K_1(\mathcal{O}_{(m,n)}(\mathbb{T})) = \mathbb{Z}_{m-1}$*

where \mathbb{Z}_k is the cyclic group $\mathbb{Z}/k\mathbb{Z}$.

Proof. Since we use the computation of K -groups in Section 4.4, we give a brief discussion. For the general Cuntz-Pimsner algebras \mathcal{O}_X associated with (A, X, ϕ) , Katsura gave the

six-term exact sequence (without the KK-theoretic method):

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\text{id}-[X]_0} & K_0(A) & \xrightarrow{\iota_*} & K_0(\mathcal{O}_X) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathcal{O}_X) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\text{id}_*-[X]_1} & K_1(A) \end{array}$$

where $\iota : A \longrightarrow \mathcal{O}_X$ is the natural inclusion and for the case where X is finitely generated by $\{u_i\}_{i=1}^n$ satisfying $\langle u_i, u_j \rangle = \delta_{i,j}$, the map $[X]_i$ are the following composition maps:

$$[X]_i : K_i(A) \xrightarrow{\phi_*} K_i(M_n(A)) \xrightarrow{\cong} K_i(A).$$

Now, we consider our algebras $\mathcal{O}_{(m,n)}$. Since $K_0(C(\mathbb{T})) = \mathbb{Z}[1]_0$, $K_1(C(\mathbb{T})) = \mathbb{Z}[z]_1$, and for the left action $\phi : C(\mathbb{T}) \longrightarrow L(X_{(m,n)}) \cong M_n(A)$,

$$\phi(1) = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \times n, \quad \phi(z) = \begin{pmatrix} 0 & z^m \\ 1_{n-1} & 0 \end{pmatrix}.$$

Hence, $[\phi(1)]_{0, M_n(C(\mathbb{T}))} = n[1]_{0, C(\mathbb{T})}$, $[\phi(z)]_{1, M_n(C(\mathbb{T}))} = m[z]_{1, C(\mathbb{T})}$. Consequently, $[X_{(m,n)}]_0$ is an n -times map and $[X_{(m,n)}]_1$ is an m -times map on \mathbb{Z} . \square

When $m \geq 2, n = 1$ for $\mathcal{O}_{(m,n)}(\mathbb{T})$, the C^* -algebras $\mathcal{O}_{(m,1)}(\mathbb{T})$ are not Kirchberg algebras: in fact, they are not simple. Moreover, these algebras are transformation group C^* -algebras on solenoid groups and they have been systematically examined by Brenken-Jørgensen [BJ], Brenken [Bre]. We recall the solenoid group. Define

$$S_m = \left\{ (x_i)_{i=0}^\infty \in \prod_{i=0}^\infty \mathbb{T} \mid x_{i+1}^m = x_i (i \in \mathbb{N}_0) \right\}.$$

Then S_m be a compact connected abelian group: it is called the *solenoid group*. We refer to [Wil2] for the solenoid. Let us define a group automorphism σ on S_m by $\sigma(x)_i = x_{i+1}$ for $x \in S_m$. Then $\mathcal{O}_{(m,1)}(\mathbb{T})$ is isomorphic to the crossed product $C(S_m) \rtimes_\sigma \mathbb{Z}$ [KW2]. For completeness, we give a brief proof for some properties on solenoid C^* -algebras examined by Brenken-Jørgensen [BJ], Brenken [Bre].

Theorem 2.2 (Brenken-Jørgensen [BJ], Brenken [Bre]).

For $m \geq 2$, the solenoid C^* -algebra $\mathcal{O}_{(m,1)}(\mathbb{T}) \cong C(S_m) \rtimes_\sigma \mathbb{Z}$ is NGCR, AF-embeddable, non-simple, residually finite dimensional.

Proof. For $x \in S_m$, define $O(x) = \{\sigma^k(x) \in S_m \mid k \in \mathbb{Z}\}$ and $D_m = \{x \in S_m \mid O(x) \text{ is dense in } S_m\}$. Then D_m is a dense set of S_m . Hence, there exist $x, y \in S_m$ such that $O(x) \neq O(y)$ and $\overline{O(x)} = S_m = \overline{O(y)}$, where $\overline{O(x)}$ is the closure set of $O(x)$ in S_m and the consequence of this observation induces the orbit space of the dynamics (S_m, σ) to not be a T_0 -topological space. Hence, $C(S_m) \rtimes_\sigma \mathbb{Z}$ is NGCR (see Section 8 of [Wil]).

For $k \geq 1$, let $\text{Per}_k(\sigma) = \{x \in S_m \mid \sigma^k(x) = x, \sigma^l(x) \neq x (1 \leq l < k)\}$ be the set of k -period points. Then for any $k \geq 1$, $\text{Per}_k(\sigma)$ is not empty, and moreover $\text{Per}(\sigma) := \bigcup_{k=1}^\infty \text{Per}_k(\sigma)$ is a countable dense set in S_m . Hence the non-wandering set of (S_m, σ) coincides with S_m . These results imply that $C(S_m) \rtimes_\sigma \mathbb{Z}$ is AF-embeddable according to the a work of

Pimsner [Pim2] and non-simple (the existence of a periodic points implies that σ is not minimal).

Let us state that $C(S_m) \rtimes_\sigma \mathbb{Z}$ is residually finite dimensional. We shall show that there is a countable family $\{\pi_n\}$ of the representations for finite dimension C^* -algebras such that $\pi := \bigoplus_n \pi_n$ is faithful. For each $x \in \text{Per}(\sigma)$, let k_x be the period of x and let us define $\rho_x : C(S_m) \longrightarrow M_{k_x}(\mathbb{C})$ by $\rho_x(f) = \text{diag}(f(x), f(\sigma(x)), \dots, f(\sigma^{k_x-1}(x)))$ and for each $z \in \mathbb{T}$, let us define a unitary $u_{x,z}$ by

$$u_{x,z} = \begin{pmatrix} 0 & z \\ 1_{k_x-1} & 0 \end{pmatrix}.$$

Then $(\rho_x, u_{x,z})$ satisfies the covariance relation, and we denote the covariance representation by $\pi_{x,z} := \rho_x \rtimes u_{x,z} : C(S_m) \rtimes_\sigma \mathbb{Z} \longrightarrow M_{k_x}(\mathbb{C})$. Set

$$\pi_z := \bigoplus_{x \in \text{Per}(\sigma)} \pi_{x,z} : C(S_m) \rtimes_\sigma \mathbb{Z} \longrightarrow \bigoplus_{x \in \text{Per}(\sigma)} M_{k_x}(\mathbb{C}).$$

Let $\{z_l\}_{l=1}^\infty$ be a dense set of \mathbb{T} . Set $\pi := \bigoplus_{l=1}^\infty \pi_{z_l}$. We shall show that π is a faithful representation.

Let $a \in C(S_m) \rtimes_\sigma \mathbb{Z}$ be a positive element such that $\pi(a) = 0$: this implies $\pi_{z_l}(a) = 0$ for $l \in \mathbb{N}$. Since $\mathbb{T} \ni z \longmapsto \pi_z(a)$ is continuous, $\pi_z(a) = 0$ for any $z \in \mathbb{T}$. For $w \in \mathbb{T}$, let us define an automorphism $\lambda_w^{(x)}$ on $M_{k_x}(\mathbb{C})$ by $\lambda_w^{(x)}(e_{ij}) = w^{i-j} e_{ij}$, where e_{ij} is the (i, j) -matrix unit. Set $\lambda_w := \bigoplus_{x \in \text{Per}(\sigma)} \lambda_w^{(x)}$. Let $\Psi : C(S_m) \rtimes_\sigma \mathbb{Z} \longrightarrow C(S_m)$ be the (canonical) faithful conditional expectation. Then

$$0 = \int_{\mathbb{T}} \lambda_z \left(\int_{\mathbb{T}} \lambda_w(\pi_z(a)) dw \right) dz = \bigoplus_{x \in \text{Per}(\sigma)} \rho_x(\Psi(a)).$$

Since $\text{Per}(\sigma)$ is dense in S_m , $\Psi(a) = 0$ and since Ψ is faithful, we can conclude that $a = 0$. Consequently, π is a faithful representation. \square

3. RIEFFEL-TYPE PROJECTION ON MATRIX ALGEBRA OVER $\mathcal{O}_{(1,2)}(\mathbb{T})$

In [Rie], Rieffel explicitly described some projections in the irrational rotation algebras A_θ and obtained the value that the trace has on them. In this section, using a similar method to [Rie], we would like to construct a projection on $M_2(\mathcal{O}_{(1,2)}(\mathbb{T}))$ which is not von Neumann equivalent to 1.

3.1. Construction of projection. We shall construct a projection of $\mathcal{O}_{(1,2)}(\mathbb{T})$. Define a $*$ -homomorphism $\phi : C(\mathbb{T}) \longrightarrow C(\mathbb{T})$ by $\phi(a)(t) = a(2t)$ for $a \in C(\mathbb{T})$, $t \in \mathbb{T} = [0, 1)$. Let us define an element on $M_2(\mathcal{O}_{(1,2)}(\mathbb{T}))$ by

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} S_1 a_1 + \phi(a_0) + a_1 S_1^* & S_2 a_1 + b_1 S_2^* \\ S_2 b_1 + a_1 S_2^* & S_1 b_1 + \phi(b_0) + b_1 S_1^* \end{pmatrix}$$

where $a_0, a_1, b_0, b_1 \in A$ are real-valued functions. Then P is a self-adjoint from the construction. We will show the following theorem in this section.

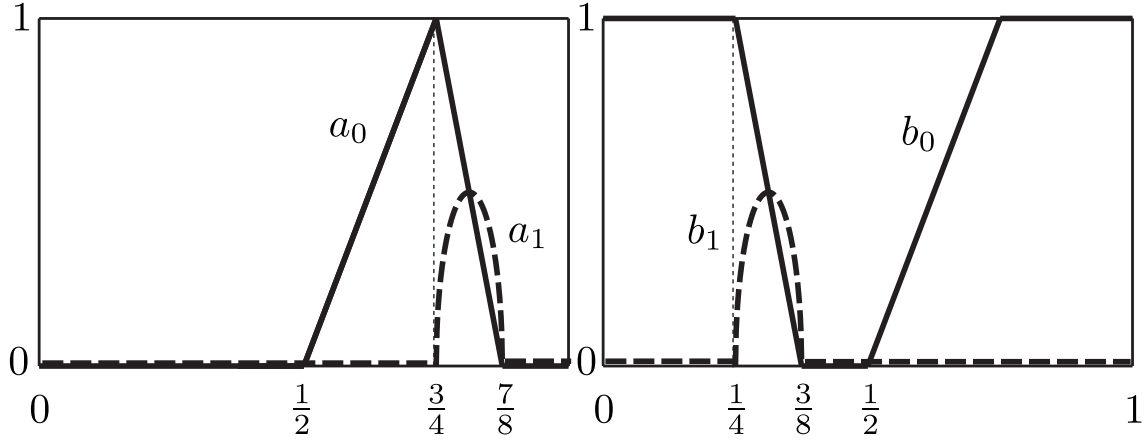


FIGURE 1. Graphs of functions a_0, a_1, b_0, b_1 .

Theorem 3.1. *Suppose that $(m, n) = (1, 2)$. Then we can construct a projection P of $M_2(\mathcal{O}_{(1,2)}(\mathbb{T}))$ that is not von Neumann equivalent to 1 and $[P]_0$ is -4 of the group $K_0(\mathcal{O}_{(1,2)}(\mathbb{T})) = \mathbb{Z}$. Let φ be the log 2-KMS state for the gauge action (see [KW]) and τ be a normalized trace on $M_2(\mathbb{C})$. Then $\varphi \otimes \tau(P) = \frac{7}{16}$.*

Proof. We need to investigate how to impose conditions in order to satisfy $P^2 = P$. First, we shall see that $P_{11} = P_{11}P_{11} + P_{12}P_{21}$. The right-side term becomes

$$\begin{aligned} P_{11}P_{11} + P_{12}P_{21} &= \phi(a_0^2) + a_1^2 + b_1^2 + \phi(a_1^2) + S_1(a_1(a_0 + \phi(a_0))) + (a_1(a_0 + \phi(a_0)))S_1^* \\ &\quad + \phi(a_1)\phi^2(a_1)S_1S_1 + \phi(a_1)\phi^2(b_1)S_2S_2 + S_1^*S_1^*\phi(a_1)\phi^2(a_1) + S_2^*S_2^*\phi(a_1)\phi^2(b_1) \end{aligned}$$

To obtain the equation $P_{11} = P_{11}P_{11} + P_{12}P_{21}$, we have to impose the following conditions:

- (1) $\phi(a_0) - \phi(a_0^2) = a_1^2 + b_1^2 + \phi(a_1^2)$
- (2) $a_1(a_0 + \phi(a_0)) = a_1$
- (3) $a_1\phi(a_1) = 0, a_1\phi(b_1) = 0$.

Let us take a_0 to be

$$a_0(t) = \begin{cases} 4(t - \frac{1}{2}) & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ 1 - 8(t - \frac{3}{4}) & \text{if } t \in [\frac{3}{4}, \frac{7}{8}] \\ 0 & \text{otherwise} \end{cases}$$

and also define a_1, b_1 by

$$a_1(t) = \begin{cases} \sqrt{\phi(a_0)(t) - \phi(a_0^2)(t)} & \text{if } t \in [\frac{3}{4}, \frac{7}{8}] \\ 0 & \text{otherwise} \end{cases} \quad b_1(t) = \begin{cases} \sqrt{\phi(a_0)(t) - \phi(a_0^2)(t)} & \text{if } t \in [\frac{1}{4}, \frac{3}{8}] \\ 0 & \text{otherwise} \end{cases}$$

We can check that these functions satisfy all the conditions.

Next, we shall check that $P_{22} = P_{21}P_{12} + P_{22}P_{22}$.

$$\begin{aligned} P_{21}P_{12} + P_{22}P_{22} &= \phi(b_0^2) + a_1^2 + b_1^2 + \phi(b_1^2) + S_1(b_1(b_0 + \phi(b_0))) + (b_1(b_0 + \phi(b_0)))S_1^* \\ &\quad + \phi(b_1)\phi^2(b_1)S_1S_1 + \phi(b_1)\phi^2(a_1)S_2S_2 + S_1^*S_1^*\phi(b_1)\phi^2(b_1) + S_2^*S_2^*\phi(b_1)\phi^2(a_1) \end{aligned}$$

To obtain the equation $P_{22} = P_{21}P_{12} + P_{22}P_{22}$, we have to impose the following conditions:

- (1) $\phi(b_0) - \phi(b_0^2) = a_1^2 + b_1^2 + \phi(b_1^2)$
- (2) $b_1(b_0 + \phi(b_0)) = b_1$
- (3) $b_1\phi(b_1) = 0, b_1\phi(a_1) = 0$.

To satisfy these conditions, we define b_0 by

$$b_0(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{4}] \\ 1 - 8(t - \frac{1}{4}) & \text{if } t \in [\frac{1}{4}, \frac{1}{2}] \\ 0 & \text{if } t \in [\frac{1}{2}, \frac{3}{4}] \\ 4(t - \frac{1}{2}) & \text{if } t \in [\frac{3}{4}, \frac{7}{8}] \\ 1 & \text{if } t \in [\frac{7}{8}, 1] \end{cases}$$

Next, we have to check that the off-diagonal part $P_{12} = P_{11}P_{12} + P_{12}P_{22}$, $P_{21} = P_{21}P_{11} + P_{22}P_{21}$ is also affirmative.

$$P_{11}P_{12} + P_{12}P_{22} = S_2a_1(a_0 + \phi(b_0)) + (\phi(a_0) + b_0)b_1S_2^*$$

and to compare the both terms, we need to impose $a_0 + \phi(b_0) = 1$ on $\text{supp}(a_1) = [3/4, 7/8]$ and $\phi(a_0) + b_0 = 1$ on $\text{supp}(b_1) = [1/4, 3/8]$. But this equation is correct from the definition of functions. Consequently we conclude that P is a projection on $M_2(\mathcal{O}_{(1,2)}(\mathbb{T}))$.

3.2. Proof of $[P]_0 = -4$ in $K_0(\mathcal{O}_{(1,2)}(\mathbb{T}))$. Let us examine $[P]_0 = -4$ in $K_0(\mathcal{O}_{(1,2)}(\mathbb{T})) = \mathbb{Z}$. We recall an exact sequence

$$0 \longrightarrow K(F(X_{(1,2)})) \longrightarrow \mathcal{T}_{(1,2)}(\mathbb{T}) \xrightarrow{\pi} \mathcal{O}_{(1,2)}(\mathbb{T}) \longrightarrow 0$$

where $K(F(X_{(1,2)}))$ is the C^* -algebra generated by one-rank operators on Fock space $F(X_{(1,2)})$ and $\mathcal{T}_{(1,2)}(\mathbb{T})$ is the Toeplitz C^* -algebra (see [Pim]). From the six-term exact sequence in the proof in Theorem 2.1, we can show that the exponential map $\delta_0 : K_0(\mathcal{O}_{(1,2)}(\mathbb{T})) \longrightarrow K_1(K(F(X_{(1,2)})))$ is a group isomorphism. Let us observe $\delta_0([P]_0)$.

There exists a projection $Q \in L(F(X_{(1,2)}))$ such that $T_{u_1}T_{u_1}^* + T_{u_2}T_{u_2}^* + Q = 1$. Set $T_i = T_{u_i}$ ($i = 1, 2$) and set an element H in $M_2(\mathcal{T}_{(1,2)}(\mathbb{T}))$ by

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} T_1a_1 + \phi(a_0) + a_1T_1^* & T_2a_1 + b_1T_2^* \\ T_2b_1 + a_1T_2^* & T_1b_1 + \phi(b_0) + b_1T_1^* \end{pmatrix}.$$

H satisfies $\pi^{(2)}(H) = P$. From the definition of δ_0 , $\delta_0[P]_0 = [\exp(2\pi iH)]_1$ in $K_1(K(F(X_{(1,2)})))$. We can check $H^2 = H - \text{diag}(\phi(a_1^2)Q, \phi(b_1^2)Q)$. Moreover, we can calculate

$$H^m = H - \text{diag}\left(\sum_{k=0}^{m-2} \phi(a_0^k)\phi(a_1^2)Q, \sum_{k=0}^{m-2} \phi(b_0^k)\phi(b_1^2)Q\right)$$

for $m \geq 2$. Let us define $\Delta_1 = \chi_{\text{supp}a_1} = \chi_{[\frac{3}{4}, \frac{7}{8}]}$, $\Delta_2 = \chi_{\text{supp}b_1} = \chi_{[\frac{1}{4}, \frac{3}{8}]}$, where χ_S is a characteristic function on $S \subset \mathbb{T}$. Then $a_1^2 = (\phi(a_0) - \phi(a_0^2))\Delta_1 = (a_0 - a_0^2)\Delta_1$ and

$$\sum_{k=0}^{m-2} a_0^k a_1^2 = \sum_{k=0}^{m-2} a_0^k (a_0 - a_0^2)\Delta_1 = (a_0 - a_0^m)\Delta_1.$$

In the same way, we can show that

$$\sum_{k=0}^{m-2} a_0^k a_1^2 = (b_0 - b_0^m)\Delta_2.$$

Hence,

$$H^m = H - \text{diag}\left(\phi((a_0 - a_0^m)\Delta_1)Q, \phi((b_0 - b_0^m)\Delta_2)Q\right)$$

for $m \geq 2$. This is also affirmative at $m = 1$. Hence,

$$\exp(2\pi i H) - 1 = \text{diag}\left((\exp(2\pi i \phi(a_0\Delta_1)) - 1)Q, (\exp(2\pi i \phi(b_0\Delta_2)) - 1)Q\right).$$

Consequently,

$$\exp(2\pi i H) = \begin{pmatrix} \exp(2\pi i \phi(a_0\Delta_1))Q & 0 \\ 0 & \exp(2\pi i \phi(b_0\Delta_2))Q \end{pmatrix} + \begin{pmatrix} 1-Q & 0 \\ 0 & 1-Q \end{pmatrix}.$$

Next, the map $K_1(A) \longrightarrow K_1(K(F(X_{(1,2)})))$ defined by

$$K_1(A) \ni [z]_1 \longmapsto [zQ + (1-Q)]_1 \in K_1(K(F(X_{(1,2)})))$$

is a group isomorphism (cf. [Pim]): hence, $\delta_0[P]_0 = [\exp(2\pi i H)]_1$ can be regarded as

$$\left[\text{diag}(\exp(2\pi i \phi(a_0\Delta_1)), \exp(2\pi i \phi(b_0\Delta_2))) \right]_1 \quad \text{in } K_1(C(\mathbb{T})).$$

From the graphs of a_0, b_0 , this element is $[z^{-4}]_1$ in $K_1(C(\mathbb{T}))$. Hence, we have finished the proof of $[P]_0 = -4$. \square

4. SUBALGEBRAS OF $\mathcal{O}_{(m,n)}(\mathbb{T})$ AND CYCLIC GROUP ACTIONS

4.1. Subalgebras of $\mathcal{O}_{(m,n)}(\mathbb{T})$. Let θ be an irrational number. Then the C^* -subalgebra $C^*(u^k, v)$ of the irrational rotation algebra A_θ generated by u^k and v is isomorphic to $A_{k\theta}$ from the relation $u^k v = e^{2\pi i k \theta} v u^k$ and from simplicity. Moreover, the subalgebra $C^*(u, v^k)$ of A_θ is isomorphic to $A_{k\theta}$. We would like to consider the corresponding problem for $\mathcal{O}_{(m,n)}(\mathbb{T})$ i.e., we shall discuss the C^* -subalgebras $C^*(z^k, S_1)$ and $C^*(z, S_1^k)$ of $\mathcal{O}_{(m,n)}(\mathbb{T})$. First, we consider some easy cases.

Lemma 4.1. *Consider $\mathcal{O}_{(m,n)}(\mathbb{T})$ for $m \geq 1, n \geq 2$ and $\gcd(m, n) = 1$. Then we have the following:*

- (1) *The C^* -subalgebra $C^*(z, S_1^k)$ of $\mathcal{O}_{(m,n)}(\mathbb{T})$ generated by z and S_1^k is isomorphic to $\mathcal{O}_{(m^k, n^k)}(\mathbb{T})$.*
- (2) *If $k|n$ (n is divided by k), then the C^* -subalgebra $C^*(z^k, S_1)$ of $\mathcal{O}_{(m,n)}(\mathbb{T})$ generated by z^k and S_1 is isomorphic to $\mathcal{O}_{(m,n)}(\mathbb{T})$.*

Proof. (1) Put $\tilde{S}_j = z^{j-1}S_1^k$ for $j = 1, \dots, n^k$. Then we can easily check the relations

$$z\tilde{S}_{n^k} = \tilde{S}_1 z^{m^k}, \quad \tilde{S}_i^* \tilde{S}_j = \delta_{ij}, \quad \sum_{i=1}^{n^k} \tilde{S}_i \tilde{S}_i^* = 1.$$

Hence, $\mathcal{O}_{(m,n)}(\mathbb{T}) \cong C^*(z, S_1^k)$.

(2) Since $k|n$, there is a $p \in \mathbb{N}$ such that $n = kl$. Hence, $z^n \in C^*(z^k, S_1)$. The relation $S_1^* z^n S_1 = z^m$ implies that $z^m \in C^*(z^k, S_1)$. Since $\gcd(m, n) = 1$, there exist $p, q \in \mathbb{Z}$ such that $mp + nq = 1$ and hence $z \in C^*(z^k, S_1)$. Since z is one of the generators of $\mathcal{O}_{(m,n)}(\mathbb{T})$, we have $C^*(z^k, S_1) \cong \mathcal{O}_{(m,n)}(\mathbb{T})$. \square

Below, we discuss for the general case of $k \in \mathbb{N}$.

Proposition 4.2. *For $m \geq 1, n \geq 2$, and $\gcd(m, n) = 1$, consider a C^* -algebra $\mathcal{O}_{(m,n)}$ generated by z and S_1 . Then the C^* -subalgebra $C^*(z^k, S_1)$ of $\mathcal{O}_{(m,n)}(\mathbb{T})$ generated by z^k and S_1 is isomorphic to $\mathcal{O}_{(m,n)}(\mathbb{T})$ for any $k \in \mathbb{N}$.*

Proof. If $\gcd(k, n) \geq 2$, then we can reduce the case of $\gcd(k, n) = 1$ using the relation $S_1^* z^n S_1 = z^m$. Let represent k, n as $k = p^\alpha p_1, n = p^\beta p_2$, where p is a prime number and p_1, p_2 do not contain p as factors. Suppose that $\alpha > \beta$ then $kp_2 = (p^{\alpha-\beta} p_1)n$ and

$$C^*(z^k, S_1) \ni S_1^* z^{kp_2} S_1 = S_1^* z^{(p^{\alpha-\beta} p_1)n} S_1 = z^{p^{\alpha-\beta} p_1 m}.$$

Since $z^{p^{\alpha-\beta} p_1 m}, z^{p^{\alpha-\beta} p_1 n} \in C^*(z^k, S_1)$ and $\gcd(m, n) = 1$, we obtain $z^{(p^{\alpha-\beta} p_1)} \in C^*(z^k, S_1)$. Repeating this process, we may assume that $\alpha \leq \beta$. Then $kp^{\beta-\alpha} p_2 = p_1 n$ and

$$C^*(z^k, S_1) \ni S_1^* z^{kp^{\beta-\alpha} p_2} S_1 = S_1^* z^{p_1 n} S_1 = z^{p_1 m}.$$

which also implies that $z^{p_1} \in C^*(z^k, S_1)$. Moreover $C^*(z^k, S_1) = C^*(z^{p_1}, S_1)$ because $(z^{p_1})^{p^\alpha} = z^k$. Continuing this process, we can reduce the case of $\gcd(k, n) = 1$. Hence, it is enough to consider the case of $\gcd(k, n) = 1$.

Since $\gcd(k, n) = 1$,

$$\{(q-1)k \pmod{n} | 1 \leq q \leq n\} = \mathbb{Z}_n.$$

Hence, for any $1 \leq q \leq n$, there exists $0 \leq l_q \leq (n-1)$ and $p_q \in \mathbb{Z}$ such that $(q-1)k = l_q + np_q$. For $1 \leq q \leq n$, put $\tilde{S}_q = z^{(q-1)k} S_1$. Then

$$\begin{aligned} \sum_{q=1}^n \tilde{S}_q \tilde{S}_q^* &= \sum_{q=1}^n (z^{qk} S_1) (z^{qk} S_1)^* = \sum_{q=1}^n (z^{l_q} S_1 z^{mp_q}) (z^{l_q} S_1 z^{mp_q})^* = \sum_{q=1}^n (z^{l_q} S_1) (z^{l_q} S_1)^* \\ &= \sum_{i=1}^n S_i S_i^* = 1 \end{aligned}$$

If we put $w := z^k$, then w is a full spectrum unitary and for $1 \leq i \leq n-1$,

$$w\tilde{S}_i = \tilde{S}_{i+1}, \quad w\tilde{S}_n = \tilde{S}_1 w^m, \quad \sum_{k=1}^n \tilde{S}_k \tilde{S}_k^* = 1.$$

Hence, $\mathcal{O}_{(m,n)}(\mathbb{T}) \cong C^*(z^k, S_1) = C^*(w, \tilde{S}_1)$ \square

4.2. Actions of cyclic groups on $\mathcal{O}_{(m,n)}(\mathbb{T})$. In [IKW], for the C^* -algebras associated complex dynamical systems, Izumi-Kajiwara-Watatani studied automorphisms arising from symmetries of the dynamical systems. In the case of $R(z) = z^n$ (whose C^* -algebra is $\mathcal{O}_{(1,n)}$), the dihedral group $\mathbb{Z}_{n-1} \rtimes \mathbb{Z}_2$ acts on the C^* -algebra $\mathcal{O}_{(1,n)}(\mathbb{T})$ (see Example 8.3 in [IKW]). They show that this dihedral action is outer. In this subsection, we shall consider extending this action on $\mathcal{O}_{(m,n)}(\mathbb{T})$. Define a $\mathbb{Z}_{|n-m|}$ -action $\beta : \mathbb{Z}_{|n-m|} \longrightarrow \text{Aut}(\mathcal{O}_{(m,n)}(\mathbb{T}))$ by

$$\beta_t : \quad z \longmapsto tz, \quad S_1 \longmapsto S_1.$$

for $t \in \mathbb{Z}_{|n-m|}$. Furthermore, we shall define a \mathbb{Z}_2 -action $\sigma \in \text{Aut}(\mathcal{O}_{(m,n)}(\mathbb{T}))$ by

$$\sigma : \quad z \longmapsto z^{-1}, \quad S_1 \longmapsto S_1.$$

We can easily check that these actions are well-defined. We shall show that these actions are outer by an elementary method.

Proposition 4.3. *Suppose that $m \geq 1, n \geq 2$ and $\gcd(m, n) = 1$. Let β, σ be group actions on $\mathcal{O}_{(m,n)}(\mathbb{T})$ defined as above.*

If $|n - m| \geq 2$, then $\mathbb{Z}_{|n-m|}$ -action β is outer. The \mathbb{Z}_2 -action σ is also outer.

Proof. First, we shall show that β is outer for $|n - m| \geq 2$. This is an well-known argument, but we give a proof for completeness. Suppose that there exists $t \in \mathbb{Z}_{|n-m|}$ with $t \neq 1$ and $u \in \mathcal{O}_{(m,n)}(\mathbb{T})$ such that $\beta_t(x) = uxu^*$. Let us represent $\mathcal{O}_{(m,n)}(\mathbb{T})$ on the Hilbert space $l^2(\mathbb{Z}[1/m])$ with CONS $\{e_k\}_{k \in \mathbb{Z}[1/m]}$ by

$$ze_k = e_{k+1}, \quad S_1 e_k = e_{\frac{n}{m}k-1}.$$

We can check that this representation is well-defined. In particular, $S_2 e_0 = e_0$. Since $\beta_t(S_2) = tS_2$, we get $tu^*e_0 = u^*e_0$, which implies that $t = 1$. This is a contradiction, so we conclude that β is outer for $|n - m| \geq 2$.

Next, we shall show that σ is outer. Suppose that there is a unitary $w \in \mathcal{O}_{(m,n)}(\mathbb{T})$ such that $\sigma(x) = wxw^*$ for $x \in \mathcal{O}_{(m,n)}(\mathbb{T})$. We consider another representation on $l^2(\mathbb{Z}[1/m])$ by

$$ze_k = e_{k+1}, \quad S_1 e_k = e_{\frac{n}{m}k}.$$

Then we shall show that w can be identified by $we_k = e_{-k}$ ($k \in \mathbb{Z}[1/m]$). Since $wS_1 = S_1w$,

$$we_0 = wS_1^k e_0 = S_1^k we_0 \in S_1^k l^2(\mathbb{Z}[1/m]) = \overline{\text{span}}\{\cdots, e_{-(n/m)^k}, e_0, e_{(n/m)^k} \cdots\}$$

for any $k \in \mathbb{N}$. This says that there exists $\lambda \in \mathbb{T}$ such that $we_0 = \lambda e_0$ since w is a unitary. Because $wz^k = z^{-k}w$ for any $k \in \mathbb{Z}$,

$$we_k = wz^k e_0 = z^{-k} we_0 = z^{-k} \lambda e_0 = \lambda e_{-k}.$$

For $p/m^q \in \mathbb{Z}[1/m]$ ($p \in \mathbb{Z}, q \in \mathbb{N}_0$), there exists $r, s \in \mathbb{Z}$ such that $p/m^q = (n/m)^q r + s$. Then

$$we_{p/m^q} = wz^s S_1^q e_r = z^{-s} S_1^q we_r = \lambda z^{-s} S_1^q e_{-r} = \lambda e_{-(n/m)^q r - s} = \lambda e_{-p/m^q}.$$

Hence, we can assume that $we_k = e_{-k}$ for $k \in \mathbb{Z}[1/m]$.

We would like to obtain a contradiction for $w \in \mathcal{O}_{(m,n)}(\mathbb{T})$. Since we supposed that

$w \in \mathcal{O}_{(m,n)}(\mathbb{T})$, there exists $K \in \mathbb{N}$, $\mu_i, \nu_i \in \mathcal{W}$, $k_i \in \mathbb{Z}$ such that

$$\left\| w - \sum_{i=1}^K S_{\mu_i} z^{k_i} S_{\nu_i}^* \right\| < 1.$$

On the other hand, for each monomial $S_{\mu_i} z^{k_i} S_{\nu_i}^*$, there exists at most one element $p_i \in \mathbb{Z}[1/m]$ such that $S_{\mu_i} z^{k_i} S_{\nu_i}^* e_{p_i} = e_{-p_i}$. Hence for $q \in \mathbb{Z}[1/m] \setminus \{p_i\}_{i=1}^K$,

$$\left\| w - \sum_{i=1}^K S_{\mu_i} z^{k_i} S_{\nu_i}^* \right\| \geq \left\| \theta_{-q, -q} \left(w - \sum_{i=1}^K S_{\mu_i} z^{k_i} S_{\nu_i}^* \right) e_q \right\| \geq \|e_{-q}\| = 1$$

where $\theta_{i,j}(\xi) = \langle \xi, e_i \rangle_{l^2(\mathbb{Z}[1/m])} e_j$ is a rank-one operator. This is a contradiction. Hence σ is outer. \square

4.3. Fixed point algebra of $\mathbb{Z}_{|n-m|}$ -action. We shall check the fixed point algebra of $\mathbb{Z}_{|n-m|}$ -action β . Let us denote $k \equiv l$ by $k = l \bmod |n-m|$.

Proposition 4.4. *Suppose that $m \geq 1, n \geq 2$, and $\gcd(m, n) = 1$. Moreover, we assume that $|n-m| \geq 2$. Then the fixed point algebra of $\mathbb{Z}_{|n-m|}$ -action β defined in Section 4.2 is the C^* -subalgebra $C^*(z^{|n-m|}, S_1)$ of $\mathcal{O}_{(m,n)}(\mathbb{T})$ generated by $z^{|n-m|}$ and S_1 , which is isomorphic to $\mathcal{O}_{(m,n)}(\mathbb{T})$ by Proposition 4.2.*

Proof. The inclusion $C^*(z^{|n-m|}, S_1) \subset \mathcal{O}_{(m,n)}(\mathbb{T})^\beta$ is trivial. We shall show that $\mathcal{O}_{(m,n)}(\mathbb{T})^\beta \subset C^*(z^{|n-m|}, S_1)$.

Since $\mathcal{O}_{(m,n)}(\mathbb{T})$ is spanned by $S_\mu z^k S_\nu^*$ ($k \in \mathbb{Z}, \mu, \nu \in \mathcal{W}_n$), we shall consider a monomial $S_\mu z^k S_\nu^*$. Suppose that $S_\mu z^k S_\nu^* \in \mathcal{O}_{(m,n)}(\mathbb{T})^\beta$: then the indices k, μ, ν satisfy

$$\sum_{i=1}^{|\mu|} (\mu_i - 1) + k - \sum_{j=1}^{|\nu|} (\nu_j - 1) \equiv 0.$$

Since $\gcd(n, |n-m|) = 1$, there is an integer p_1 such that $\gamma_1 := (\mu_1 - 1) + p_1 n \equiv 0$. Moreover, for $\mu_2 - 1 - p_1 m \in \mathbb{Z}$, there is an integer p_2 such that $\gamma_2 := (\mu_2 - 1 - p_1 m) + p_2 n \equiv 0$. Repeating this process, we find that there are $\{p_i\}_{i=2}^{|\mu|}$ such that $\gamma_i := (\mu_i - 1 - p_{i-1} m) + p_i n \equiv 0$ ($i = 2, \dots, |\mu|$). Hence,

$$\begin{aligned} S_\mu &= z^{\mu_1-1} S_1 z^{\mu_2-1} S_1 \dots z^{\mu_{|\mu|}-1} S_1 \\ &= z^{\mu_1-1+p_1 n} S_1 z^{(\mu_2-1-p_1 m)+p_2 n} S_1 \dots z^{(\mu_{|\mu|}-1-p_{|\mu|-1} m+p_{|\mu|} n} S_1 z^{-p_{|\mu|} m} \\ &= z^{\gamma_1} S_1 z^{\gamma_2} S_1 \dots z^{\gamma_{|\mu|}} S_1 z^{-p_{|\mu|} m} \end{aligned}$$

Furthermore, there are integers $\{q_i\}_{i=1}^{|\nu|}$ such that $\delta_1 := (\nu_1 - 1) + p_1 n \equiv 0$ and $\delta_i := (\nu_i - 1 - p_{i-1} m) + p_i n \equiv 0$ ($i = 2, \dots, |\nu|$). Then

$$S_\nu = z^{\delta_1} S_1 z^{\delta_2} S_1 \dots z^{\delta_{|\nu|}} S_1 z^{-q_{|\nu|} m}.$$

Hence, we obtain

$$S_\mu z^k S_\nu^* = z^{\gamma_1} S_1 z^{\gamma_2} S_1 \dots z^{\gamma_{|\mu|}} S_1 z^{-p_{|\mu|} m + k + q_{|\nu|} m} S_1^* z^{-\delta_{|\nu|}} \dots S_1^* z^{-\delta_2} S_1^* z^{-\delta_1}.$$

Then

$$\begin{aligned}
0 &\equiv \sum_{i=1}^{|\mu|} \gamma_i - \sum_{i=1}^{|\nu|} \delta_i \\
&= \left(\sum_{i=1}^{|\mu|} (\mu_i - 1) - \sum_{i=1}^{|\nu|} (\nu_i - 1) \right) + \sum_{i=1}^{|\mu|-1} p_i(n-m) - \sum_{i=1}^{|\nu|-1} q_i(n-m) + p_{|\mu|}n - q_{|\nu|}n \\
&\equiv -k + p_{|\mu|}m - q_{|\nu|}m.
\end{aligned}$$

Hence, we obtain $\gamma_i, \delta_j, -p_{|\mu|}m + k + q_{|\nu|}m \in |n-m|\mathbb{Z}$ and this implies that $S_\mu z^k S_\nu \in C^*(z^{|n-m|}, S_1)$. Hence we conclude that $\mathcal{O}_{(m,n)}(\mathbb{T})^\beta \subset C^*(z^{|n-m|}, S_1)$. \square

4.4. Fixed point algebra of symmetric action. We shall determine the fixed point algebra of the \mathbb{Z}_2 -action σ defined in Section 4.2. In this subsection, we shall show the following proposition:

Proposition 4.5. *Suppose that $m \geq 1, n \geq 2$, and $\gcd(m, n) = 1$. Let σ be the \mathbb{Z}_2 -action on $\mathcal{O}_{(m,n)}(\mathbb{T})$ defined in Section 4.2. Then the fixed point algebra $\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma$ is a Kirchberg algebra satisfying UCT and its K -group is following: if m is even, then*

$$K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma) = \mathbb{Z}_{n-1}, \quad K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma) = 0, \quad [1]_0 = 0$$

and if m is odd, then

$$K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma) = \mathbb{Z}[1]_0 \oplus \mathbb{Z}_{n-1} \oplus \mathbb{Z}_{n-1}, \quad K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma) = \mathbb{Z}.$$

Proof. First, we compute the K -groups of $\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma$. Since $\mathcal{O}_{(m,n)}(\mathbb{T})$ is simple and σ is outer, $\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma$ is Morita equivalent to $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2$. Hence the group $K_i(\mathcal{O}_{(m,n)}(\mathbb{T})^\sigma)$ is isomorphic to $K_i(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2)$ ($i = 0, 1$). If w is the unitary of $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\sigma \mathbb{Z}_2$ implementing σ , then elements in the crossed product have the form $x + yw$ where $x, y \in \mathcal{O}_{(m,n)}(\mathbb{T})$. To compute K -groups, we construct another Cuntz-Pimsner algebra. We define a \mathbb{Z}_2 -action σ_0 on $A_0 = C(\mathbb{T})$ by $\sigma_0(z_0) = z_0^{-1}$ (where z_0 is the unitary generator of A_0) and define $B = A_0 \rtimes_{\sigma_0} \mathbb{Z}_2$ with implement unitary w_0 . Let $Y = B^{\oplus n}$ with the right B -action defined by $(b_i)_{i=1}^n \cdot b = (b_i b)_{i=1}^n$ for $(b_i)_{i=1}^n \in Y, b \in B$. Let us put $u'_i = (0, \dots, 1, \dots, 0) \in Y$ for $i = 1, \dots, n$. Define a B -valued inner product by

$$\langle (b_i)_i, (b'_i)_i \rangle_B = \sum_{i=1}^n b_i^* b'_i.$$

Then Y is a full Hilbert B -module with this inner product. Let us define a left B -action $\phi : B \longrightarrow L_B(Y) \cong M_n(B)$ by

$$\phi(z_0) = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \middle| \begin{array}{c} z_0^m \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \right), \quad \phi(w_0) = \left(\begin{array}{c|ccc} w_0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & w_0 z_0^m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & w_0 z_0^m & \cdots & 0 \end{array} \middle| \begin{array}{c} 0 \\ w_0 z_0^m \\ 0 \\ \vdots \\ 0 \end{array} \right)$$

Then we can check that $\phi(w_0)$ is self-adjoint unitary and that $\phi(w_0)\phi(z_0)\phi(w_0) = \phi(z_0)^{-1}$. Hence, ϕ is a $*$ -homomorphism. In fact, ϕ is faithful:

Lemma 4.6. ϕ is faithful.

Proof. Define E to be the canonical faithful conditional expectation from B onto A_0 . Let $\widehat{\sigma}_0$ be the dual action of σ_0 and let u be the implement unitary of $B \rtimes_{\widehat{\sigma}_0} \mathbb{Z}_2$. Define a unitary in $M_n(B \rtimes_{\widehat{\sigma}_0} \mathbb{Z}_2)$ by $U = \text{diag}(u, \dots, u)$. Then we can check that $\text{Ad}(U)(\phi(z_0)) := U\phi(z_0)U^* = \phi(z_0)$ and $\text{Ad}(U)(\phi(w_0)) = -\phi(w_0)$. Set $E_1 := \frac{1}{2}(\text{id} + \text{Ad}(U)) : \phi(B) \longrightarrow \phi(B)$, Then E_1 is a faithful conditional expectation onto $\phi(A_0)$ and satisfies $E_1 \circ \phi = \phi \circ E$. Since $\phi|_{A_0} : A_0 \longrightarrow \phi(A_0)$ is an isomorphism, the equation $E_1 \circ \phi = \phi \circ E$ induces the faithfulness of ϕ . \square

Hence, we can construct the Cuntz-Pimsner algebra \mathcal{O}_Y from these data. Then $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2$ is isomorphic to \mathcal{O}_Y by both universalities; the isomorphism is determined by

$$\psi(z) = z_0, \psi(w) = w_0, \psi(S_i) = S_{u'_i} \ (i = 1, \dots, n).$$

Next, let us compute the K -groups of \mathcal{O}_Y instead of those of $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2$. From the six-term exact sequence of Cuntz-Pimsner algebra, we obtain the exact sequence

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{\text{id}_* - [Y]_0} & K_0(B) & \xrightarrow{\iota_*} & K_0(\mathcal{O}_Y) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(\mathcal{O}_Y) & \xleftarrow{\iota_*} & K_1(B) & \xleftarrow{\text{id}_* - [Y]_1} & K_1(B) \end{array}$$

where $[Y]_i$ ($i = 0, 1$) is the group homomorphism arising from ϕ .

We recall that $K_0(B) = \mathbb{Z}^3$ with generators $e_0 := [1]_0, e_1 := [\frac{1}{2}(1 + w_0)]_0, e_2 := [\frac{1}{2}(1 + w_0 z_0)]_0$ and $K_1(B) = 0$. We can show

$$[Y]_0(e_0) = ne_0, \quad [Y]_0(e_1) = e_1 + (n-1) \left[\frac{1}{2}(1 + w_0 z_0^m) \right]_0, \quad [Y]_0(e_2) = n \left[\frac{1}{2}(1 + w_0 z_0^m) \right]_0$$

(see the proof of Theorem 2.1). Since

$$\left[\frac{1}{2}(1 + w_0 z_0^m) \right]_0 = \begin{cases} \left[\frac{1}{2}(1 + w_0) \right]_0 & m: \text{ even} \\ \left[\frac{1}{2}(1 + w_0 z_0) \right]_0 & m: \text{ odd}, \end{cases}$$

if m is even, then $[Y]_0(e_1) = ne_1, [Y]_0(e_2) = ne_1$ and if m is odd, then $[Y]_0(e_1) = e_1 + (n-1)e_2$ and $[Y]_0(e_2) = ne_2$. From the six-term exact sequence, we can conclude that

$$K_0(\mathcal{O}_Y) = \begin{cases} \mathbb{Z}_{n-1}e_0 & m: \text{ even} \\ \mathbb{Z}_{n-1}e_0 \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}_{n-1}e_2 & m: \text{ odd} \end{cases}, \quad K_1(\mathcal{O}_Y) = \begin{cases} 0 & m: \text{ even} \\ \mathbb{Z} & m: \text{ odd}, \end{cases}$$

Since the isomorphism from $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma})$ to $K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2)$ is defined by $[p]_0 \longmapsto [\frac{1}{2}(p + pw)]_0$, we have determined the K -groups of $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ in Proposition 4.5.

Since σ is outer (Proposition 4.3) and $\mathcal{O}_{(m,n)}(\mathbb{T})$ is purely infinite simple, $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is purely infinite simple (by using Lemma 10 of [KK] and $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is a hereditary algebra of $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2$). Since $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is a hereditary algebra of $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2$ and $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\sigma} \mathbb{Z}_2$ is nuclear, $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is also nuclear. The separability of $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is trivial. Moreover \mathcal{O}_Y satisfies UCT because B satisfies UCT, and $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ is Morita equivalent to \mathcal{O}_Y , so $\mathcal{O}_{(m,n)}(\mathbb{T})^{\sigma}$ also satisfies UCT. Hence, we have completed the proof of Proposition 4.5. \square

5. ENTROPY ESTIMATE FOR THE CANONICAL ENDOMORPHISM ON $\mathcal{O}_{(1,n)}(\mathbb{T})$

For the Cuntz algebra \mathcal{O}_n , Choda has computed Voiculescu's entropy ([Cho]) for the *Cuntz's canonical endomorphism* defined by

$$\Phi_0(x) = \sum_{i=1}^n S_i x S_i^*, \quad (x \in \mathcal{O}_n).$$

In this section, we consider an analogy to this problem for $\mathcal{O}_{(1,n)}(\mathbb{T})$

For $\gcd(m, n) = 1$ and $n \geq 2$, let us define the *canonical endomorphism* on $\mathcal{O}_{(m,n)}$ by

$$\Phi(x) = \sum_{i=1}^n S_i x S_i^*, \quad (x \in \mathcal{O}_{(m,n)}(\mathbb{T})).$$

Its name is derived from one of the Cuntz algebras \mathcal{O}_n . Then Φ is a *-endomorphism on $\mathcal{O}_{(m,n)}$. We would like to compute Voiculescu's topological entropy for Φ on $\mathcal{O}_{(1,n)}(\mathbb{T})$. Our method is similar to that of Boca-Goldstein([BG])

Let us recall the definition of the Voiculescu's topological entropy([Voi]). Let B be a nuclear C^* -algebra with unit. Let $CPA(B)$ be the triples (ϕ, ψ, C) , where C is a finite-dimensional C^* -algebra, and $\phi : B \rightarrow C$ and $\psi : C \rightarrow B$ are unital completely positive maps. Let $Pf(B)$ be the set of finite subsets of B . For an $\omega \in Pf(B)$, put

$$rcp(\omega; \delta) = \inf \{ \text{rank } C : (\phi, \psi, C) \in CPA(B), \|\psi \circ \phi(a) - a\| < \delta, a \in B \}$$

where $\text{rank } C$ means the dimension of a maximal abelian self-adjoint subalgebra of C . Since B is nuclear, for any $\omega \in Pf(B)$ and $\delta > 0$, there exists $(\phi, \psi, C) \in CPA(B)$ such that $\|\psi \circ \phi(a) - a\| < \delta, a \in \omega$. For a unital *-endomorphism β of B , put

$$ht(\beta, \omega; \delta) = \limsup_{N \rightarrow \infty} \frac{1}{N} \log rcp(\omega \cup \beta(\omega) \cup \dots \cup \beta^{N-1}(\omega); \delta)$$

and

$$ht(\beta; \omega) = \sup_{\delta > 0} ht(\beta, \omega; \delta).$$

Then (Voiculescu's) *topological entropy* $ht(\beta)$ of β is defined by

$$ht(\beta) = \sup_{\omega \in Pf(B)} ht(\beta, \omega).$$

We recall the Kolmogorov-Sinai type theorem.

Theorem 5.1 (Voiculescu [Voi]). *Let $\omega_j \in Pf(B)$ such that $\omega_1 \subset \omega_2 \subset \dots$ and the linear span of $\bigcup_{j \in \mathbb{N}} \omega_j$ is dense in B . Then*

$$ht(\beta) = \sup_{j \in \mathbb{N}} ht(\beta, \omega_j).$$

Let φ be a state of B with $\varphi \circ \beta = \varphi$. An estimate between $ht(\beta)$ and *Connes-Narnhofer-Thirring (CNT) entropy* $h_\varphi(\beta)$ ([CNT]) is given by

$$h_\varphi(\beta) \leq ht(\beta).$$

which was proved by Voiculescu([Voi]).

The C^* -algebra $\mathcal{O}_{(1,n)}(\mathbb{T})$ has exactly one $\log n$ -KMS state φ for the gauge action of $\mathcal{O}_{(1,n)}$. This KMS-state is written as $\varphi = \tau \circ E$, where τ is the unique normalized trace

on the $(1, n)$ -type Bunce-Deddens algebra and E is the conditional expectation onto the Bunce-Deddens algebra. Our main theorem in this section is as follows:

Theorem 5.2. *Suppose $m = 1$ and $n \geq 2$. Let φ be the unique $\log n$ -KMS state for gauge action of $\mathcal{O}_{(1,n)}(\mathbb{T})$. Let Φ be the canonical endomorphism defined as above. Then the Voiculescu's topological entropy $ht(\Phi)$ for Φ and the CNT-entropy $h_\varphi(\Phi)$ for Φ and φ are both equal to $\log n$;*

$$h_\varphi(\Phi) = ht(\Phi) = \log n.$$

Proof. First, we define a map $\rho_r : \mathcal{O}_{(1,n)}(\mathbb{T}) \longrightarrow M_{n^r}(\mathbb{C}) \otimes \mathcal{O}_{(1,n)}(\mathbb{T})$ for $r \geq 1$ by

$$\rho_r(x) = \sum_{|\mu|, |\nu|=r} e_{\mu\nu} \otimes S_\mu^* x S_\nu.$$

We can check that this map is $*$ -homomorphism and induce the isomorphism between $\mathcal{O}_{(1,n)}(\mathbb{T})$ and $M_{n^r}(\mathbb{C}) \otimes \mathcal{O}_{(1,n)}(\mathbb{T})$.

For $\mu \in \mathcal{W}_n^{(k)}$, we can see μ as $\sum_{i=1}^k (\mu_i - 1)n^{i-1}$ via $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. Define

$$\|\mu\|_k = \sum_{i=1}^k (\mu_i - 1)n^{i-1}$$

for $\mu \in \mathcal{W}_n^{(k)}$.

Lemma 5.3. *Let $N \geq 1$ and assume that $|\alpha|, |\beta| \leq s$ and $N + s \leq r$ and $1 \leq l \leq N$. Then for $|k| \leq n^s$,*

$$\rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) = \begin{cases} x_0 \otimes z^{q_0} + x_1 \otimes z^{q_0+1} & |\alpha| = |\beta| \\ \sum_{|\eta|=|\alpha|-|\beta|} \left(y_{0,\eta} \otimes S_\eta z^{q_0} + y_{1,\eta} \otimes S_\eta z^{q_0+1} \right) & |\alpha| > |\beta|, k \geq 0 \\ \sum_{|\eta|=|\alpha|-|\beta|} \left(y_{0,\eta} \otimes z^{q_0} S_\eta + y_{1,\eta} \otimes z^{q_0+1} S_\eta \right) & |\alpha| > |\beta|, k \leq 0 \\ \sum_{|\eta|=|\alpha|-|\beta|} \left(y_{0,\eta} \otimes S_\eta^* z^{q_0} + y_{1,\eta} \otimes S_\eta^* z^{q_0+1} \right) & |\alpha| < |\beta|, k \geq 0 \\ \sum_{|\eta|=|\alpha|-|\beta|} \left(y_{0,\eta} \otimes z^{q_0} S_\eta^* + y_{1,\eta} \otimes z^{q_0+1} S_\eta^* \right) & |\alpha| < |\beta|, k \leq 0 \end{cases}$$

where $x_0, x_1, y_{0,\eta}, y_{1,\eta}$ are partial isometries that depend on α, β, k , and $|q_0| \leq n^s$

Proof. We consider the case of $k \geq 0$ (the case of $k \leq 0$ is similar) and suppose that $|\beta| \leq |\alpha|$.

$$\begin{aligned} & \rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) \\ &= \sum_{|\mu|, |\nu|=r} \sum_{|\gamma|=l} e_{\mu,\nu} \otimes S_\mu^* S_\gamma (S_\alpha z^k S_\beta^*) S_\gamma^* S_\nu = \sum_{|\mu|=r-l-|\alpha|, |\nu|=r-l-|\beta|} \left(\sum_{|\gamma|=l} e_{\mu\alpha\gamma, \nu\beta\gamma} \right) \otimes S_\mu^* z^k S_\nu \end{aligned}$$

Let us put $x_{\mu,\nu} = \sum_{|\gamma|=l} e_{\mu\alpha\gamma,\nu\beta\gamma}$, and $p = r - l - |\beta|$. Then

$$\begin{aligned}
\rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) &= \sum_{|\mu|=r-l-|\alpha|, |\nu|=p} x_{\mu,\nu} \otimes S_\mu^* z^k S_\nu \\
&= \sum_{|\mu|=r-l-|\alpha|} \left(\sum_{\{\nu|0 \leq \|\nu\|_p \leq n^p-k-1\}} x_{\mu,\nu} \otimes S_\mu^* z^k S_\nu + \sum_{q=1}^{n^{s-p}} \sum_{\{\nu|qn^p-k \leq \|\nu\|_p \leq (q+1)n^p-k-1\}} x_{\mu,\nu} \otimes S_\mu^* z^k S_\nu \right) \\
&= \sum_{|\mu|=r-l-|\alpha|} \left(\sum_{\{\eta|k \leq \|\eta\|_p \leq n^p-1\}} x_{\mu,\eta} \otimes S_\mu^* S_\eta + \sum_{q=1}^{n^{(s-p)}} \sum_{\{\eta|qn^p \leq \|\eta\|_p \leq (q+1)n^p-1\}} x_{\mu,\eta} \otimes S_\mu^* S_\eta z^q \right)
\end{aligned}$$

Let us define

$$\mathcal{V}_0 = \{\eta \in \mathcal{W}_n^{(p)} | k \leq \|\eta\|_p \leq n^p - 1\}, \quad \mathcal{V}_q = \{\eta \in \mathcal{W}_n^{(p)} | qn^p \leq \|\eta\|_p \leq (q+1)n^p - 1\}$$

for $1 \leq q \leq n^{s-p}$. Then there exists $0 \leq q_0 \leq n^{(s-p)}$ such that all \mathcal{V}_q are empty except $q = q_0$ or $q = q_0 + 1$. We shall define $x_{\mu,\eta} = 0$ for $\|\eta\|_p < k, n - p + k \leq \|\eta\|_p$.

$$\rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) = \sum_{|\mu|=r-l-|\alpha|} \left(\sum_{\eta \in \mathcal{V}_{q_0}} x_{\mu,\eta} \otimes S_\mu^* S_\eta z^{q_0} + \sum_{\eta \in \mathcal{V}_{q_0+1}} x_{\mu,\eta} \otimes S_\mu^* S_\eta z^{q_0+1} \right)$$

If we assume $|\alpha| = |\beta|$, then

$$\rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) = \left(\sum_{|\mu|=r-l-|\alpha|} x_{\mu,\eta(\mu)} \right) \otimes z^{q_0} + \left(\sum_{|\mu|=r-l-|\alpha|} x_{\mu,\eta(\mu)} \right) \otimes z^{q_0+1},$$

and $\sum_{|\mu|=r-l-|\alpha|} x_{\mu,\eta(\mu)}$ is a partial isometry. If $|\alpha| > |\beta|$, then

$$\rho_r \circ \Phi^l(S_\alpha z^k S_\beta^*) = \sum_{|\eta|=|\alpha|-|\beta|} \left(x_{\mu(\eta),\eta} \otimes S_\eta z^{q_0} + x_{\mu(\eta),\eta} \otimes S_\eta z^{q_0+1} \right),$$

where $x_{\mu(\eta),\eta}$ are partial isometries.

If we take the involution, we get the case of $|\alpha| < |\beta|$. □

Let us define

$$\omega(s) = \{S_\alpha z^k S_\beta^* | |\beta|, |\alpha| \leq s, |k| \leq n^s\}$$

which is increasing for $s \in \mathbb{N}$ and the linear span of the union is dense in $\mathcal{O}_{(1,n)}(\mathbb{T})$. Since $\mathcal{O}_{(1,n)}(\mathbb{T})$ is nuclear, there exist unital completely positive maps $\phi_0 : \mathcal{O}_{(1,n)}(\mathbb{T}) \longrightarrow M_R(\mathbb{C})$ and $\psi_0 : M_R(\mathbb{C}) \longrightarrow \mathcal{O}_{(1,n)}(\mathbb{T})$ such that

$$\begin{aligned}
\sum_{|q| \leq n^s} & \left(\|\psi_0 \circ \phi_0(S_\eta z^q) - S_\eta z^q\| + \|\psi_0 \circ \phi_0(z^q S_\eta) - z^q S_\eta\| \right. \\
& \left. + \|\psi_0 \circ \phi_0(z^q S_\eta^*) - z^q S_\eta^*\| + \|\psi_0 \circ \phi_0(S_\eta^* z^q) - S_\eta^* z^q\| \right) < \frac{\delta}{n^s}
\end{aligned}$$

for $0 \leq |\eta| \leq s, |q| \leq n^s$. Let us define $\phi : \mathcal{O}_{(1,n)}(\mathbb{T}) \longrightarrow M_R(\mathbb{C}) \otimes M_{n^r}(\mathbb{C})$ and $\psi : M_R(\mathbb{C}) \otimes M_{n^r}(\mathbb{C}) \longrightarrow \mathcal{O}_{(1,n)}(\mathbb{T})$ by

$$\phi = (\text{id} \otimes \phi_0) \circ \rho_r, \quad \psi = \rho_r^{-1} \circ (\text{id} \otimes \psi_0).$$

where $r = s + N$. Then for $S_\alpha z^k S_\beta^* \in \omega(s)$, we can show that

$$\|\psi \circ \phi \circ \Phi^l(S_\alpha z^k S_\beta^*) - \Phi^l(S_\alpha z^k S_\beta^*)\| < \delta$$

for $0 \leq l \leq N - 1$. Hence,

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{1}{N} \log rcp(\omega(s) \cup \Phi(\omega(s)) \cup \dots \cup \Phi^{N-1}(\omega(s)); \delta) \\ & \leq \limsup_{N \rightarrow \infty} \frac{1}{N} \log(Rn^r) = \limsup_{N \rightarrow \infty} \frac{1}{N} (\log R + (s + N) \log n) = \log n \end{aligned}$$

and, using Theorem 5.1, we have finished the proof of $ht(\Phi) \leq \log n$.

On the other hand, we shall show that $\log n \leq h_\varphi(\Phi)$. Using the gauge action of $\mathcal{O}_{(1,n)}(\mathbb{T})$, we can take the conditional expectation onto the Bunce-Deddens algebra $\mathcal{B}_{(1,n)}$. Moreover, we consider \mathbb{T} -action on $\mathcal{B}_{(1,n)}$ defined as follow. First, we construct \mathbb{T} -action on $M_k(C(\mathbb{T}))$ which are building blocks of $\mathcal{B}_{(1,n)}$. For $t \in \mathbb{T}$, let us define $\gamma_t^{(k)} : M_k(C(\mathbb{T})) \rightarrow M_k(C(\mathbb{T}))$ by

$$\gamma_t^{(k)}(f)(z) = U_t^{(k)} f(t^k z) U_t^{(k)*} \quad (f \in M_k(C(\mathbb{T})))$$

where $U_t^{(k)}$ is the unitary of $M_k(\mathbb{C})$ defined by $U_t^{(k)} = \text{diag}(1, t, \dots, t^{(k-1)})$. Then these actions are compatible for the inductive limit system of $\mathcal{B}_{(1,n)}$, so we can construct the action arising from $\gamma^{(n^k)}$'s; we shall denote it by γ . Then we can check that the fixed point algebra $\mathcal{B}_{(1,n)}^\gamma$ is the continuous functions $C(K_n)$, where K_n is the Cantor set, which is the maximal abelian algebra of Cuntz algebra. Hence, we obtain a conditional expectation onto $C(K_n)$ from $\mathcal{B}_{(1,n)}$ (and also from $\mathcal{O}_{(1,n)}(\mathbb{T})$). Moreover, $\Phi|_{C(K_n)}$ is the canonical shift on K_n and $\varphi|_{C(K_n)}$ is the canonical trace, so we obtain $h_{\varphi|_{C(K_n)}}(\Phi|_{C(K_n)}) = \log n$. Hence,

$$\log n = h_{\varphi|_{C(K_n)}}(\Phi|_{C(K_n)}) \leq h_\varphi(\Phi) \leq ht(\Phi) \leq \log n.$$

Consequently $h_\varphi(\Phi) = ht(\Phi) = \log n$. □

6. DUAL ACTION OF THE GAUGE ACTION AND K -THEORY

In [Mat], Matsumoto investigated the dual action of the gauge action on C^* -algebras associated with a subshift on the level of the K -groups to study dimension groups for the subshift. We follow his argument for $\mathcal{O}_{(m,n)}(\mathbb{T})$. Here, we compute the behavior of the dual action on K -groups.

Let $\alpha : \mathbb{T} \rightarrow \mathcal{O}_{(m,n)}(\mathbb{T})$ be the canonical gauge action and consider the crossed product $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$, which is the universal C^* -algebra generated by the $*$ -algebra $L^1(\mathbb{T}, \mathcal{O}_{(m,n)}(\mathbb{T}))$ whose multiplication and involution are defined as follows:

$$f * g(t) = \int_{\mathbb{T}} f(s) \alpha_s(g(s^{-1}t)) ds, \quad f^*(t) = \alpha_t(f(t^{-1})^*).$$

for $f, g \in L^1(\mathbb{T}, \mathcal{O}_{(m,n)}(\mathbb{T}))$, $t \in \mathbb{T}$. Let $\hat{\alpha}$ be the *dual action* of α which is defined at the level of functions by $\hat{\alpha}(f)(t) = tf(t)$. The crossed product $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z}$ is stably isomorphic to $\mathcal{O}_{(m,n)}(\mathbb{T})$. Let $p_0 : \mathbb{T} \rightarrow \mathcal{O}_{(m,n)}(\mathbb{T})$ be the constant function whose value everywhere is the unit of $\mathcal{O}_{(m,n)}(\mathbb{T})$. By [Ro], the fixed point algebra $\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha$ is isomorphic to the algebra $p_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})p_0$. The isomorphism between them is given

by the correspondence $j : \mathcal{O}_{(m,n)}(\mathbb{T})^\alpha \ni x \mapsto \widehat{x} \in L^1(\mathbb{T}, \mathcal{O}_{(m,n)}(\mathbb{T})) \subset \mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$ where the function \widehat{x} is defined by $\widehat{x}(t) = x$ for $t \in \mathbb{T}$.

Lemma 6.1. *The projection p_0 is full in $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$.*

Proof. The proof of this lemma is the same as that of Lemma 4.1 of [Mat], but we give it for completeness. Suppose that there exists a nondegenerate representation π of $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$ such that $\pi(p_0) = 0$. For any $x \in \mathcal{O}_{(m,n)}(\mathbb{T})$,

$$\widehat{x} * p_0(t) = \int_{\mathbb{T}} \widehat{x}(s) \alpha_s(p_0(s^{-1}t)) ds = x.$$

Hence, $\widehat{x} * p_0 = \widehat{x}$. This implies that $\widehat{x} \in \ker \pi$. For $x \in \mathcal{O}_{(m,n)}(\mathbb{T})$, $|\mu| = k \in \mathbb{N}$,

$$\widehat{xS_\mu} * \widehat{S_\mu^*}(t) = \int_{\mathbb{T}} \widehat{xS_\mu}(s) \alpha_s(\widehat{S_\mu^*}(s^{-1}t)) ds = xS_\mu \int_{\mathbb{T}} \alpha_s(\alpha_{s^{-1}t}(S_\mu^*)) ds = xS_\mu \alpha_t(S_\mu^*) = t^{-k} xS_\mu S_\mu^*$$

and we take the summation for the words of length k , $(\sum_{|\mu|=k} \widehat{xS_\mu} * \widehat{S_\mu^*})(t) = t^{-k}x$. We can also show that $\widehat{xS_\mu^*} * \widehat{S_\mu}(t) = t^kx$ for $|\mu| = k$. Hence any $\mathcal{O}_{(m,n)}(\mathbb{T})$ -valued function of the form $\mathbb{T} \ni t \mapsto t^kx$ is contained in the ideal $\ker(\pi)$. This implies that p_0 is a full projection in $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$. \square

Since p_0 is a full projection in $\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}$, there is an isometry $v \in M((\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \otimes \mathbb{K})$ such that $v^*v = 1 \otimes 1$, $vv^* = p_0 \otimes 1$ and

$$\text{Ad}(v^*) : p_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})p_0 \otimes \mathbb{K} \longrightarrow (\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \otimes \mathbb{K}$$

induce an isomorphism. We shall show that we can treat $\text{Ad}(v^*)$ as an inclusion map $\iota : p_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})p_0 \otimes \mathbb{K} \longrightarrow (\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \otimes \mathbb{K}$ in K -groups.

Lemma 6.2. $K_i(\iota) = K_i(\text{Ad}(v^*))$ for $i = 0, 1$.

Proof. From Proposition 12.2.2 of [Bla], we can take a continuous path of isometries $(w_t)_{t \in (0,1]}$ in the multiplier algebra $M((\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \otimes \mathbb{K})$ such that $w_t w_t^* \longrightarrow 0$ ($t \rightarrow 0$) strictly. Put $v_t = w_t v w_t^* + (1 - w_t w_t^*)$ for $t \in (0, 1]$ and $v_0 = 1$. Then $(\text{Adv}_t^*(x))_{t \in [0,1]}$ for $x \in p_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})p_0 \otimes \mathbb{K}$ is the norm continuous path i.e., $\text{Ad}(v^*)$ and ι are homotopy equivalent. Hence, the above path implies that $K_i(\iota) = K_i(\text{Ad}(v^*))$ for $i = 0, 1$. \square

First we consider the K_0 -group. The group $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha)$ is isomorphic to $\mathbb{Z}[1/n]$ and for any $k \in \mathbb{N}$, $[S_1^k S_1^{k*}]_0$ corresponds to $1/n^k$. Note that $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \cong K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})$ by the induced map $\psi_0 = K_0(\iota \circ j) : [q]_0 \mapsto [\widehat{q}]_0$. Let us define a map $\beta_0 : K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \longrightarrow K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha)$ by $\beta_0 = \psi_0^{-1} \circ K_0(\widehat{\alpha}) \circ \psi_0$:

$$\begin{array}{ccc} K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) & \xrightarrow{K_0(\widehat{\alpha})} & K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \\ \psi_0 \uparrow \cong & & \psi_0 \uparrow \cong \\ K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) & \xrightarrow{\beta_0} & K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha). \end{array}$$

Lemma 6.3. *For any projection $q \in \mathcal{B}_{(m,n)} = \mathcal{O}_{(m,n)}(\mathbb{T})^\alpha$, $\beta_0[q]_0 = [S_1 q S_1^*]_0$.*

Proof. This proof is the same as Lemma 4.5 of [Mat], but for convenience, we repeat it. It is enough to show that $K_0(\widehat{\alpha})[\widehat{q}]_0 = [\widehat{S_1 q S_1^*}]_0$ in $K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T})$. Since $q \in \mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}$, we have

$$\begin{aligned} (\widehat{S_1} * \widehat{q})(t) &= \int_{\mathbb{T}} \widehat{S_1}(s) \alpha_s(\widehat{q}(s^{-1}t)) ds = S_1 \int_{\mathbb{T}} \alpha_s(q) ds = S_1 q \\ (\widehat{S_1} * \widehat{q} * \widehat{S_1}^*)(t) &= \int_{\mathbb{T}} (\widehat{S_1} * \widehat{q})(s) \alpha_s(\widehat{S_1}^*(s^{-1}t)) ds = S_1 q \alpha_t(S_1^*) = t^{-1} S_1 q S_1^*. \end{aligned}$$

Hence, $\widehat{\alpha}(\widehat{S_1} * \widehat{q} * \widehat{S_1}^*)(t) = S_1 q S_1^*$ and this implies that $\widehat{\alpha}(\widehat{S_1} * \widehat{q} * \widehat{S_1}^*) = \widehat{S_1 q S_1^*}$. We can easily check that $\widehat{S_1}^* \widehat{S_1} = p_0$. Put $W = \widehat{S_1} * \widehat{q}$; then $W^* * W = \widehat{q} * \widehat{S_1}^* * \widehat{S_1} * \widehat{q} = \widehat{q} * p_0 * \widehat{q} = \widehat{q}$, $W * W^* = \widehat{S_1} * \widehat{q} * \widehat{S_1}^*$. This implies that

$$K_0(\widehat{\alpha})[\widehat{q}]_0 = K_0(\widehat{\alpha})[\widehat{S_1} * \widehat{q} * \widehat{S_1}^*]_0 = [\widehat{S_1 q S_1^*}]_0 \quad \text{in } K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes \mathbb{T}).$$

Hence, the proof is complete. \square

Next, we consider the K_1 -group. We remark that $K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}) = \mathbb{Z}[1/m]$ and $[S_1^k z S_1^{*k} + 1 - S_1^k S_1^{*k}]_1$ corresponds to $1/m^k$. The map $\psi_1 := K_1(\iota \circ j) : K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}) \longrightarrow K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T})$ is determined by $\psi_1[S_1^k z S_1^{*k} + 1 - S_1^k S_1^{*k}]_1 = [(S_1^k z S_1^{*k} + 1 - S_1^k S_1^{*k})^{\wedge} + 1 - p_0]_1$, where 1 is the unit of the unitization C^* -algebra $(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T})^{\dagger}$. Let us define $\beta_1 : K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}) \longrightarrow K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha})$ by $\beta_1 = \psi_1^{-1} \circ K_1(\widehat{\alpha}) \circ \psi_1$;

$$\begin{array}{ccc} K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T}) & \xrightarrow{K_1(\widehat{\alpha})} & K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T}) \\ \psi_1 \uparrow \cong & & \psi_1 \uparrow \cong \\ K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}) & \xrightarrow{\beta_1} & K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}). \end{array}$$

We recall the following lemma (Lemma 1.2 of [Cu]).

Lemma 6.4. *Let B be a C^* -algebra. Then for any partial isometry $s \in B^{\dagger}$ and unitary $u \in s^* s B^{\dagger} s^* s$,*

$$[u + 1 - s^* s]_1 = [sus^* + 1 - ss^*]_1 \text{ in } K_1(B).$$

Let us check that β_1 is the $1/m$ -times map at the level of the K_1 -group. It is enough to calculate for $[z]_1$.

Lemma 6.5. $\beta_1[z]_1 = \frac{1}{m}[z]_1$ in $K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}) = \mathbb{Z}[1/m]$.

Proof. We compute $K_1(\widehat{\alpha})[\widehat{z} + 1 - p_0]_1$ in $K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_{\alpha} \mathbb{T})$ instead of $\beta_1[z]_1$. From Lemma 6.4, $[\widehat{z} + (1 - p_0)]_1 = [\widehat{S_1} * \widehat{z} * \widehat{S_1}^* + 1 - \widehat{S_1} * \widehat{S_1}^*]_1$. By a similar calculation to the one in Proposition 6.3,

$$\begin{aligned} \widehat{\alpha}^{\dagger}(\widehat{S_1} * \widehat{z} * \widehat{S_1}^* + 1 - \widehat{S_1} * \widehat{S_1}^*) &= \widehat{S_1 z S_1^*} + 1 - \widehat{S_1 S_1^*} = \widehat{S_1 z S_1^*} + p_0 - \widehat{S_1 S_1^*} + (1 - p_0) \\ &= (S_1 z S_1^* + 1 - S_1 S_1^*)^{\wedge} + (1 - p_0). \end{aligned}$$

This implies that

$$\beta_1[z]_1 = [S_1 z S_1^* + (1 - S_1 S_1^*)]_1 \quad \text{in } K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^{\alpha}).$$

Note that since $S_1 \notin \mathcal{O}_{(m,n)}(\mathbb{T})^\alpha$, we cannot apply Lemma 6.4. The element $S_1 z S_1^* + (1 - S_1 S_1^*)$ is $\text{diag}(z, 1_{n-1})$ in $\mathcal{B}_{(m,n)}$ and

$$\left[\begin{pmatrix} z & 0 \\ 0 & 1_{n-1} \end{pmatrix} \right]_1 = \frac{1}{m} \left[\begin{pmatrix} 0 & z^m \\ 1_{n-1} & 0 \end{pmatrix} \right]_1 = \frac{1}{m} [z]_1.$$

Hence, $\beta_1[z]_1 = \frac{1}{m}[z]_1$. □

We summarize these lemmas below.

Theorem 6.6. Suppose that $m \geq 1, n \geq 1$, and $\gcd(m, n) = 1$. Let α be the gauge action of $\mathcal{O}_{(m,n)}(\mathbb{T})$ and $\hat{\alpha}$ be the dual action of α . For $i = 0, 1$, let $\psi_i : K_i(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \longrightarrow K_i(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})$ be the group isomorphism defined as above. Let $K_i(\hat{\alpha})$ be the induced map of the dual action $\hat{\alpha}$ on $K_i(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T})$ and define $\beta_i := \psi_i \circ K_i(\hat{\alpha}) \circ \psi_i^{-1}$. Then β_0 is a $1/n$ -times map on $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \cong \mathbb{Z}[1/n]$ and β_1 is a $1/m$ -times map on $K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \cong \mathbb{Z}[1/m]$.

We shall give another proof of the computation of K -groups of $\mathcal{O}_{(m,n)}(\mathbb{T})$ (see also Proposition 2.1). When we apply the Pimsner-Voiculescu six-term exact sequence for $\mathcal{O}_{(m,n)}(\mathbb{T}) \otimes \mathbb{K} = \mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T} \rtimes_{\hat{\alpha}} \mathbb{Z}$, we obtain the following exact sequence:

$$\begin{array}{ccccc} K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) & \xrightarrow{\text{id} - K_0(\hat{\alpha}^{-1})} & K_0(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) & \longrightarrow & K_0(\mathcal{O}_{(m,n)}(\mathbb{T})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_{(m,n)}(\mathbb{T})) & \longleftarrow & K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) & \xleftarrow{\text{id} - K_1(\hat{\alpha}^{-1})} & K_1(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \end{array}$$

Since $K_i(\mathcal{O}_{(m,n)}(\mathbb{T}) \rtimes_\alpha \mathbb{T}) \cong K_i(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha)$ ($i = 0, 1$) and from the above argument, we have

$$\begin{array}{ccccc} K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) & \xrightarrow{\text{id} - \beta_0^{-1}} & K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) & \longrightarrow & K_0(\mathcal{O}_{(m,n)}(\mathbb{T})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_{(m,n)}(\mathbb{T})) & \longleftarrow & K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) & \xleftarrow{\text{id} - \beta_1^{-1}} & K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) \end{array}$$

From $K_0(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) = K_0(\mathcal{B}_{(m,n)}) = \mathbb{Z}[1/n]$, $K_1(\mathcal{O}_{(m,n)}(\mathbb{T})^\alpha) = K_1(\mathcal{B}_{(m,n)}) = \mathbb{Z}[1/m]$ and Theorem 6.6, β_0 and β_1 correspond to the multiplication of $1/n$ and $1/m$, respectively. From an easy calculation, we obtain another computation of K -groups of $\mathcal{O}_{(m,n)}(\mathbb{T})$.

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